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Open string amplitudes in various gauges

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Abstract: Recently, Schnabl constructed the analytic solution of the open string tachyon. Subsequently, the absence of the physical states at the vacuum was proved. The development relies heavily on the use of the gauge condition different from the ordinary one. It was shown that the choice of gauge simplifies the analysis drastically. In this paper we perform the calculation of the amplitudes in Schnabl gauge and find that the off-shell amplitudes is still complicated. To find simple off-shell amplitudes, we choose different gauges for the states and propagators. In particular, we propose a modified Schnabl gauge for the propagator and show that this gauge choice simplifies the calculation of the off-shell amplitudes. We also show that this modified use of the propagator reproduces the on-shell four point amplitudes correctly. We apply this method to open superstring field theory.

Keywords: String Field Theory, Bosonic Strings, Superstrings and Heterotic Strings.

Contents

1. Introduction

One of the recent striking achievements in string field theory is the analytic proof of Sen's conjectures [1, 2]. In [3], Schnabl constructed an analytic solution for the equation of motion

$$Q_B\Phi + \Phi * \Phi = 0, \tag{1.1}$$

in Witten's cubic string field theory [4] and proved that the height of the tachyon potential at the vacuum is related to the tension of the D-brane [1]. The consistency of the solutions has been checked in [5, 6]. Subsequently, the Sen's third conjecture which states that there is no physical state at this vacuum was proved analytically [7].

The equation of motion (1.1) is a highly non-linear equation with an infinite numbers of degrees of freedom. In the Siegel gauge $b_0\Phi = 0$, which is traditionally used for the most of the computation, the equation can be solved by tedious numerical calculation such as level truncation [8 − 10]. In this gauge, the calculations of the amplitudes are also formidable task [11, 12].

Recent developments of the string field theory rely heavily on the use of the proper gauge for the calculation. Schnabl realized that the gluing rule of string field theory does not match with the Siegel gauge and used another gauge which is more useful in the star operation [3]. Subsequent proof of the absence of the physical degree of freedom also relies heavily on the use of this gauge [7]. In [13], the technique has been generalized to obtain solutions for a ghost number zero string field equation.

Another problem in the string field theory is the complicated expression of off-shell amplitudes [11, 12]. Recent developments suggest that this gauge simplifies the analysis of the amplitudes. However, the propagator in this gauge turns out not to be convenient for explicit calculation which was stated in [3].

In this paper, we consider the mixed use of the gauge choice for the states and the propagators. We use the Siegel gauge for the states and propose a novel gauge for the propagators. We show that this gauge choice gives a simple formula for the four point off-shell amplitudes in Witten's cubic string field theory. This modified gauge can also be applied to WZW-like action [14] of open superstring field theory. Although the physical meaning of the mixed gauge choice is not clear at this stage, the different choice of the gauge is so useful for computing the on-shell amplitudes in the string field theory.

This paper is organized as follows. In the next section, we review the star calculus in $\tilde{z}$ coordinate and how to calculate four point amplitudes. In section 3, we compute the expression of four-tachyon off-shell amplitudes in the Schnabl gauge. In section 4, we propose the use of the modified Schnabl gauge for the propagator and apply this method to some four point amplitudes. In section 5, we show how to use this modified gauge in WZW-like action of the superstring to compute the string amplitudes. In section 6, we compute the four point amplitude for tachyons in GSO($-$) sector and the effective quartic terms of the gauge fields in the zero momentum limit. The final section is devoted to some discussions.

2. Witten's cubic interaction in $\tilde{z}$ coordinate and amplitudes

In Witten's open string field theory, the gluing condition simplifies in the coordinates $\tilde{z} = \arctan z$. In this coordinate, the primary field $\phi(z)$ of dimension h is given by [3]

$$\tilde{\phi}(\tilde{z}) = \left(\frac{dz}{d\tilde{z}}\right)^h \phi(\tan\tilde{z}) = (\cos\tilde{z})^{-2h}\phi(\tan\tilde{z}). \tag{2.1}$$

The scaling generator can be written by the energy momentum tensor in this coordinate as

$$\mathcal{L}_0 = \oint \frac{d\tilde{z}}{2\pi i}\tilde{z}T_{\tilde{z}\tilde{z}}(\tilde{z}) = L_0 + \sum_{k=0}^{\infty}\frac{2(-1)^{k+1}}{4k^2-1}L_{2k}. \tag{2.2}$$

The scaling operator U_r can be defined as

$$U_r = \left(\frac{2}{r}\right)^{\mathcal{L}_0}. \tag{2.3}$$

The action of U_r with the field of conformal dimension h is simply given by

$$U_r\tilde{\phi}(\tilde{z})U_r^{-1} = \left(\frac{2}{r}\right)^h \tilde{\phi}\left(\frac{2}{r}\tilde{z}\right). \tag{2.4}$$

A non-trivial property of this operator is

$$e^{-\beta\widehat{\mathcal{L}}_0} = U_{2+2\beta}^{\dagger} U_{2+2\beta}, \tag{2.5}$$

where

$$\widehat{\mathcal{L}}_0 = \mathcal{L}_0 + \mathcal{L}_0{}^{\dagger}. \tag{2.6}$$

Because of this property, it seems convenient to define an operator

$$\widehat{U}_r = U_r^{\dagger} U_r. \tag{2.7}$$

We can easily find $\widehat{U}_2 = 1$ and the product rule is

$$\widehat{U}_r \widehat{U}_s = \widehat{U}_{r+s-2}. \tag{2.8}$$

Three vertex of string field theory defines a mapping gluing two fields into one state. In the coordinate system, the mapping can be simply given by

$$\tilde{\phi}_1(0)|0\rangle * \tilde{\phi}_2(0)|0\rangle = \widehat{U}_3 \tilde{\phi}_1\left(\frac{\pi}{4}\right)\tilde{\phi}_2\left(-\frac{\pi}{4}\right)|0\rangle. \tag{2.9}$$

The BPZ conjugate of states is defined by the conformal transformation $I(z) = -1/z$. For example, the conjugate of the above state is given by

$$\mathrm{bpz}\left(\widehat{U}_3 \tilde{\phi}_1\left(\frac{\pi}{4}\right)\tilde{\phi}_2\left(-\frac{\pi}{4}\right)|0\rangle\right) = (-1)^{|\phi_1||\phi_2|}\langle 0| I\circ\tilde{\phi}_2\left(-\frac{\pi}{4}\right) I\circ\tilde{\phi}_1\left(\frac{\pi}{4}\right)\widehat{U}_3, \tag{2.10}$$

where $|\phi|$ is the Grassman parity of ϕ. In the $\tilde{z}$ coordinates inversion I acts simply as a translation $I\circ\tilde{\phi}(\tilde{z}) = \tilde{\phi}(\tilde{z}+\pi/2) = \tilde{\phi}(\tilde{z}-\pi/2)$. More generally, the gluing of the states of the form $\widehat{U}\tilde{\phi}$ takes a simple form

$$\widehat{U}_r\tilde{\phi}_1(\tilde{x})|0\rangle * \widehat{U}_s\tilde{\phi}_2(\tilde{y})|0\rangle = \widehat{U}_{r+s-1}\tilde{\phi}_1\left(\tilde{x}+\frac{\pi}{4}(s-1)\right)\tilde{\phi}_2\left(\tilde{y}-\frac{\pi}{4}(r-1)\right)|0\rangle. \tag{2.11}$$

In order to obtain the exact solutions, Schnabl used a gauge [3]

$$\mathcal{B}_0\Phi = 0, \tag{2.12}$$

where $\mathcal{B}_0$ is the zero mode of the b ghost in the $\tilde{z}$ coordinate

$$\mathcal{B}_0 = \oint \frac{d\tilde{z}}{2\pi i}\tilde{z}b(\tilde{z}) = b_0 + \sum_{k=0}^{\infty}\frac{2(-1)^{k+1}}{4k^2-1}b_{2k}. \tag{2.13}$$

Its anti-commutator with BRST charge is given by $\{Q_B, \mathcal{B}_0\} = \mathcal{L}_0$. For later convenience, we define

$$\widehat{\mathcal{B}}_0 = \mathcal{B}_0 + \mathcal{B}_0^{\dagger}, \tag{2.14}$$

from which we find a relation $\{Q_B, \widehat{\mathcal{B}}_0\} = \widehat{\mathcal{L}}_0$. The advantage of the Schnabl gauge is that the form of the fields including ghosts and $\widehat{\mathcal{L}}_0, \widehat{\mathcal{B}}_0$ close under the star operations and the

JHEP01(2007)011

action of the BRST charge [3]. This choice turns out to be crucial for obtaining exact solution of eq. (1.1).

Useful identities including $\widehat{\mathcal{B}}_0$ are

$$
\begin{aligned}
\widehat{\mathcal{B}}_0 \widehat{U}_r &= \widehat{U}_r \widehat{\mathcal{B}}_0, &(2.15)\\
\widehat{\mathcal{B}}_0 U_r^\dagger &= \frac{2}{r} U_r^\dagger \widehat{\mathcal{B}}_0, &(2.16)\\
\widehat{\mathcal{B}}_0(\phi_1 * \phi_2) &= \frac{\pi}{2}(-1)^{\mathrm{gh}(\phi_1)}(B_1\phi_1) * \phi_2 + \phi_1 * (\widehat{\mathcal{B}}_0 \phi_2), &(2.17)
\end{aligned}
$$

where $B_1 = b_1 + b_{-1}$. We find that anti-commutators

$$
\begin{aligned}
\{\mathcal{B}_0, \tilde{c}(\tilde{z})\} &= \tilde{z},\\
\{B_1, \tilde{c}(\tilde{z})\} &= 1, \qquad (2.18)
\end{aligned}
$$

are also useful for the calculation of the amplitudes. Correlation functions of the fields in the $\tilde{z}$ coordinates are summarized as

$$
\begin{aligned}
\langle \partial \tilde{X}^\mu(\tilde{z}) \partial \tilde{X}^\nu(\tilde{w})\rangle &= -\frac{\alpha'}{2}\eta^{\mu\nu}\frac{1}{\sin^2(\tilde{z}-\tilde{w})},\\
\langle \tilde{c}(\tilde{x})\tilde{c}(\tilde{y})\tilde{c}(\tilde{w})\rangle &= \sin(\tilde{x}-\tilde{y})\sin(\tilde{y}-\tilde{w})\sin(\tilde{w}-\tilde{x}). \qquad (2.19)
\end{aligned}
$$

In this section, we will show how the above relations will be used to obtain the amplitudes.

Witten's cubic action is given by

$$
S = -\frac{1}{g^2}\left[\frac{1}{2}\langle \Phi, Q_B \Phi\rangle + \frac{1}{3}\langle \Phi, \Phi * \Phi\rangle\right]. \qquad (2.20)
$$

To find the effective action, we will use the background field method [15]. We separate the field into the background field ϕ_b and quantum fluctuation R as

$$
\Phi = \phi_b + R. \qquad (2.21)
$$

We consider the path integral of the field R. The contribution of R to the action is

$$
S = -\frac{1}{g^2}\left[\frac{1}{2}\langle R, Q_B R\rangle + \langle R, \phi_b * \phi_b\rangle + \langle \phi_b, R * R\rangle + \frac{1}{3}\langle R, R * R\rangle\right]. \qquad (2.22)
$$

To find the effective action, we need to consider the propagator of the quantum fluctuation field R and fix the gauge. We can obtain the four point process by shifting the field R [15]

$$
R \to R - \mathcal{P}\phi_b * \phi_b, \qquad (2.23)
$$

where $\mathcal{P}$ is the propagator which satisfies a relation

$$
Q_B \mathcal{P} = 1. \qquad (2.24)
$$

As a result of the shift of the quantum fluctuation field R, we find that the quartic interaction term is given in $\tilde{z}$ coordinates as

$$
A_4 = \frac{1}{2g^2}\left\langle I \circ \tilde{\phi}_b\left(\frac{\pi}{4}\right) I \circ \tilde{\phi}_b\left(-\frac{\pi}{4}\right)\widehat{U}_3 \mathcal{P} \widehat{U}_3 \tilde{\phi}_b\left(\frac{\pi}{4}\right)\tilde{\phi}_b\left(-\frac{\pi}{4}\right)\right\rangle. \qquad (2.25)
$$

Here we assumed that the field $\tilde{\phi}_b$ has ghost number 1 and changed the order of two conjugated fields to cancel the overall -1 factor.

3. Four point amplitudes in the Schnabl gauge

We are now going to compute the amplitudes in the Schnabl gauge. Let us consider the fields of the form $\phi_b = c(z)V_1(z)$. For example, the tachyon and photon vertices are simply given by $\phi_b = c(z)e^{ikX(z)}$ and $\phi_b = \epsilon_\mu(k)c(z)\sqrt{\frac{2}{\alpha'}}\partial X^\mu(z)e^{ikX(z)}$. Note that the tachyon and the photon are the same in Siegel gauge and in Schnabl gauge.

In the Schnabl gauge (2.12), the propagator is given by

$$\mathcal{P} = \frac{\mathcal{B}_0}{\mathcal{L}_0} Q_B \frac{\mathcal{B}_0^\dagger}{\mathcal{L}_0{}^\dagger}. \tag{3.1}$$

For the computation of the amplitude in this gauge, we need two Schwinger parameters for this propagator

$$\begin{aligned}\mathcal{P} &= \mathcal{B}_0 \int_0^\infty dt_1 e^{-t_1\mathcal{L}_0} Q_B \mathcal{B}_0^\dagger \int_0^\infty dt_2 e^{-t_2\mathcal{L}_0^\dagger} \\ &= \int_0^\infty dt_1 \int_0^\infty dt_2 \mathcal{B}_0 U_{T_1} Q_B \mathcal{B}_0^\dagger U_{T_2}^\dagger,\end{aligned} \tag{3.2}$$

where $T_1 = 2e^{t_1}$, $T_2 = 2e^{t_2}$.

Using the commutation rules, we find

$$\widehat{U}_3 \mathcal{B}_0 U_{T_1} Q_B \mathcal{B}_0^\dagger U_{T_2}^\dagger \widehat{U}_3 = \widehat{U}_3 \mathcal{B}_0 U_{T_1} \widehat{U}_3 - \widehat{U}_3 \mathcal{B}_0 U_{T_1} \mathcal{B}_0^\dagger U_{T_2}^\dagger \widehat{U}_3 Q_B. \tag{3.3}$$

Thus in the Schnabl gauge, the four point amplitude (2.25) is written as follows

$$\begin{aligned}A_4 =& \frac{1}{2g^2}\int_0^\infty dt \left\langle I\circ\tilde{\phi}_1\left(\frac{\pi}{4}\right) I\circ\tilde{\phi}_2\left(-\frac{\pi}{4}\right) \widehat{U}_3\mathcal{B}_0 U_T \widehat{U}_3 \tilde{\phi}_3\left(\frac{\pi}{4}\right)\tilde{\phi}_4\left(-\frac{\pi}{4}\right)\right\rangle \\ &-\frac{1}{2g^2}\int_0^\infty dt_1 \int_0^\infty dt_2 \left\langle I\circ\tilde{\phi}_1\left(\frac{\pi}{4}\right) I\circ\tilde{\phi}_2\left(-\frac{\pi}{4}\right) \widehat{U}_3\mathcal{B}_0 U_{T_1}\mathcal{B}_0^\dagger U_{T_2}^\dagger \widehat{U}_3 Q_B \tilde{\phi}_3\left(\frac{\pi}{4}\right)\tilde{\phi}_4\left(-\frac{\pi}{4}\right)\right\rangle.\end{aligned} \tag{3.4}$$

In the case that all $\phi_i{}'s$ are primary field with weight h_i, using

$$\widehat{U}_3\mathcal{B}_0 U_T \widehat{U}_3 = U^\dagger_{\frac{3T+2}{T}}\left(\frac{3T}{3T+2}\mathcal{B}_0 - \frac{2}{3T+2}\mathcal{B}_0^\dagger\right) U_{\frac{3T+2}{2}}, \tag{3.5}$$

and eq. (2.4), the correlator in the first term can be simplified as

$$\begin{aligned}&\left\langle I\circ\tilde{\phi}_1\left(\frac{\pi}{4}\right) I\circ\tilde{\phi}_2\left(-\frac{\pi}{4}\right) U_3^\dagger U_3 \mathcal{B}_0 U_{T_1} U_3^\dagger U_3 \tilde{\phi}_3\left(\frac{\pi}{4}\right)\tilde{\phi}_4\left(-\frac{\pi}{4}\right)\right\rangle \\ &\qquad = \left\langle \tilde{\phi}_1(\tilde{z}_1)\tilde{\phi}_2(-\tilde{z}_1)\left(\frac{3T_1}{3T_1+2}\mathcal{B}_0 - \frac{2}{3T_1+2}\mathcal{B}_0^\dagger\right)\tilde{\phi}_3(\tilde{z}_2)\tilde{\phi}_4(-\tilde{z}_2)\right\rangle \\ &\qquad\quad \times\left(\frac{2T_1}{3T_1+2}\right)^{h_1+h_2}\left(\frac{4}{3T_1+2}\right)^{h_3+h_4},\end{aligned} \tag{3.6}$$

where

$$\tilde{z}_1 = \frac{\pi(2T_1+1)}{3T_1+2}, \quad \tilde{z}_2 = \frac{\pi}{3T_1+2}. \tag{3.7}$$

As an example, let us consider the four point amplitude of tachyons. Substituting into the tachyon vertex operator, the first term of eq. (3.4) yields

$$\begin{aligned}&\frac{1}{2g^2}(2\pi)^{26}\delta^{26}(\textstyle\sum k_i)\displaystyle\int_0^{1/2}dy\; y^{-2-\alpha' s}(1-y)^{-2-\alpha' u}\\&\quad\times|\sin(2\tilde{z}_1)|^{2-\alpha'(k_1^2+k_2^2)}|\sin(2\tilde{z}_2)|^{2-\alpha'(k_3^2+k_4^2)}|\sin(\tilde{z}_1+\tilde{z}_2)|^{4-\alpha'(k_1^2+k_2^2+k_3^2+k_4^2)}\\&\quad\times|\sin(\tilde{z}_1-\tilde{z}_2)|^{4-\alpha'(k_1^2+k_2^2+k_3^2+k_4^2)}\left(\frac{2T_1}{3T_1+2}\right)^{\alpha'(k_1^2+k_2^2)-2}\left(\frac{4}{3T_1+2}\right)^{\alpha'(k_3^2+k_4^2)-2},\end{aligned}\tag{3.8}$$

where new variable y is introduced such as

$$\begin{aligned}&y=-\frac{\sin(2\tilde{z}_1)\sin(2\tilde{z}_2)}{\sin^2(\tilde{z}_1-\tilde{z}_2)},\quad 1-y=\frac{\sin^2(\tilde{z}_1+\tilde{z}_2)}{\sin^2(\tilde{z}_1-\tilde{z}_2)},\\&y=1/2\quad(\text{for } t=0),\quad y=0\quad(\text{for } t=\infty).\end{aligned}\tag{3.9}$$

The second term of (3.4) which vanishes for on-shell amplitude is rather complicated. Using

$$U_3^\dagger U_3\mathcal{B}_0U_{T_1}\mathcal{B}_0^\dagger U_{T_2}^\dagger U_3^\dagger U_3=-\frac{4}{3T_1+3T_2-4}U^\dagger_{\frac{3T_1+3T_2-4}{T_1}}\mathcal{B}_0^\dagger\mathcal{B}_0U_{\frac{3T_2+3T_1-4}{T_2}},\tag{3.10}$$

and evaluating correlator, we get

$$\begin{aligned}&-\frac{1}{2g^2}\int_0^\infty dt_1\int_0^\infty dt_2\frac{4(-\alpha' k_3^2-\alpha' k_4^2+2)}{3T_1+3T_2-4}\left(\frac{2}{V}\right)^{-2+\alpha'(k_1^2+k_2^2)}\left(\frac{2}{W}\right)^{-2+\alpha'(k_3^2+k_4^2)}\\&\times(2\pi)^{26}\delta^{26}(\textstyle\sum k_i)\displaystyle\left|\sin\left(\frac{\pi}{V}\right)\right|^{2\alpha' k_1\cdot k_2}\left|\sin\left(\frac{\pi}{W}\right)\right|^{2\alpha' k_3\cdot k_4}\left|\cos\left(\frac{\pi}{2V}-\frac{\pi}{2W}\right)\right|^{2\alpha'(k_1\cdot k_3+k_2\cdot k_4)}\\&\times\left|\cos\left(\frac{\pi}{2V}+\frac{\pi}{2W}\right)\right|^{2\alpha'(k_1\cdot k_4+k_2\cdot k_3)}\frac{\pi}{2V}\left(\cos\left(\frac{\pi}{V}\right)+\cos\left(\frac{\pi}{W}\right)\right)\left(\frac{\pi}{W}\cos\left(\frac{\pi}{W}\right)-\sin\left(\frac{\pi}{W}\right)\right),\end{aligned}\tag{3.11}$$

where

$$\frac{2}{V}=\frac{2T_1}{3(T_1+T_2)-4},\quad\frac{2}{W}=\frac{2T_2}{3(T_1+T_2)-4}.\tag{3.12}$$

4. Four point amplitudes of tachyons and gauge fields in the modified Schnabl gauge

In the previous section, we have seen that the four point amplitudes in the Schnabl gauge is very complicated. The complication stems from the form of the propagator in this gauge. To avoid these difficulties, we will use the propagator in the gauge $\widehat{\mathcal{B}}_0R=0$ not in the Schnabl gauge $\mathcal{B}_0R=0$ for the quantum fluctuation field.

Let us consider the four point amplitudes in this modified Schnabl gauge. In this gauge, the propagator can be written as

$$\mathcal{P}=\frac{\widehat{\mathcal{B}}_0}{\widehat{\mathcal{L}}_0},\tag{4.1}$$

which is manifestly self-conjugate and commutes with $\widehat{U}$ which appears in the amplitudes. We find that the quartic interaction term is given by

$$A_4 = \frac{1}{2g^2}\Big\langle I \circ \tilde{\phi}_1\left(\frac{\pi}{4}\right) I \circ \tilde{\phi}_2\left(-\frac{\pi}{4}\right) \widehat{U}_3 \frac{\widehat{\mathcal{B}}_0}{\widehat{\mathcal{L}}_0} \widehat{U}_3 \tilde{\phi}_3\left(\frac{\pi}{4}\right) \tilde{\phi}_4\left(-\frac{\pi}{4}\right)\Big\rangle. \tag{4.2}$$

We will use Schwinger parametrization as

$$\frac{\widehat{\mathcal{B}}_0}{\widehat{\mathcal{L}}_0} = \int_0^\infty d\beta \widehat{\mathcal{B}}_0 e^{-\beta \widehat{\mathcal{L}}_0} = \int_0^\infty d\beta \widehat{\mathcal{B}}_0 \widehat{U}_{2\beta+2}. \tag{4.3}$$

Using the manipulation rule $\widehat{U}_r \widehat{U}_s = \widehat{U}_{r+s-2}$ and the fact that $\widehat{\mathcal{B}}_0$ and $\widehat{U}_r$ commute with each other, we easily find

$$A_4 = \frac{1}{2g^2}\int_0^\infty d\beta \left\langle I \circ \tilde{\phi}_1\left(\frac{\pi}{4}\right) I \circ \tilde{\phi}_2\left(-\frac{\pi}{4}\right) \widehat{\mathcal{B}}_0 \widehat{U}_{2\beta+4} \tilde{\phi}_3\left(\frac{\pi}{4}\right) \tilde{\phi}_4\left(-\frac{\pi}{4}\right)\right\rangle. \tag{4.4}$$

By expressing $\widehat{U}_{2\beta+4} = U^\dagger_{2\beta+4} U_{2\beta+4}$ and relations (2.4) and (2.16), we can move $U^\dagger_{2\beta+4}$ to the left and $U_{2\beta+4}$ to the right to find

$$\begin{aligned} A_4 = \frac{1}{2g^2}\int_0^\infty \frac{d\beta}{(\beta+2)^{\sum_i h_i+1}} \Big\langle I \circ \tilde{\phi}_1\left(\frac{\pi}{4}\left(\frac{1}{\beta+2}\right)\right) I \circ \tilde{\phi}_2\left(-\frac{\pi}{4}\left(\frac{1}{\beta+2}\right)\right) \\ \times \widehat{\mathcal{B}}_0 \tilde{\phi}_3\left(\frac{\pi}{4}\left(\frac{1}{\beta+2}\right)\right) \tilde{\phi}_4\left(-\frac{\pi}{4}\left(\frac{1}{\beta+2}\right)\right)\Big\rangle. \end{aligned} \tag{4.5}$$

By changing variable by $t = 1/(\beta+2)$, we arrive at the following formula of four point amplitudes;

$$A_4 = \frac{1}{2g^2}\int_0^{1/2} dt\, t^{\sum_i h_i - 1} \left\langle I \circ \tilde{\phi}_1\left(\frac{\pi}{4}t\right) I \circ \tilde{\phi}_2\left(-\frac{\pi}{4}t\right) \left(\mathcal{B}_0 + \mathcal{B}_0^\dagger\right) \tilde{\phi}_3\left(\frac{\pi}{4}t\right) \tilde{\phi}_4\left(-\frac{\pi}{4}t\right)\right\rangle. \tag{4.6}$$

Let us apply this formula to the four point tachyon amplitudes. Here, we use the Siegel gauge for the states instead of $\widehat{\mathcal{B}}_0$ gauge. Although the physical meaning of the off-shell amplitudes in this mixed gauge choice is not clear, it can be proved that the on-shell four point amplitudes are reproduced correctly.

Suppose we have modified b_0 as

$$b_0' = b_0 + \sum_{n=-\infty, n\neq 0}^{\infty} a_n b_n \tag{4.7}$$

with some parameters a_n. When we use a gauge for the internal state $b_0' R = 0$, the propagator of this gauge is given by

$$\mathcal{P}' = \frac{b_0'}{L_0'}, \tag{4.8}$$

where

$$L_0' = \{Q_B, b_0'\} = L_0 + \sum_{n=-\infty, n\neq 0}^{\infty} a_n L_n. \tag{4.9}$$

We will compute the variation of the propagator with respect to a_n;

$$\frac{\partial \mathcal{P}}{\partial a_n} = b_n \frac{1}{L_0'} - b_0' \frac{1}{L_0'} L_n \frac{1}{L_0'}. \tag{4.10}$$

Because of the relations $L_0' = \{Q_B, b_0'\}$ and $L_n = \{Q_B, b_n\}$, this expression can be rewritten as

$$\frac{\partial \mathcal{P}}{\partial a_n} = \{Q_B, b_0' \frac{1}{L_0'} b_n \frac{1}{L_0'}\}. \tag{4.11}$$

Eq. (4.11) implies that the dependence on the parameters is BRST exact and decouples from the correlation functions of BRST closed states. This statement may be an extension of the propagator of the usual gauge fields. That is to say, even though the propagator of gauge fields contains the gauge parameters, the total amplitudes do not depend on the parameters. The equation (4.11) also implies that we can use the propagator whose gauge condition is different from the one for external states. The equation (4.11) states that the difference is just BRST exact and the amplitudes for BRST invariant (on-shell) states coincide with the on-shell amplitudes in the Siegel gauge.

The tachyon vetex operators are $\phi_i = ce^{ik_i\cdot X}$, and the following correlator included in the formula (4.6) is easily evaluated by using eq. (D.12) from ref. [3] and typical correlation functions for $e^{ik\cdot X}$ operators

$$\begin{aligned} &\left\langle I \circ \tilde{c}e^{ik_1\cdot\tilde{X}}\left(\frac{\pi}{4}t\right) I \circ \tilde{c}e^{ik_2\cdot\tilde{X}}\left(-\frac{\pi}{4}t\right)\left(\mathcal{B}_0 + \mathcal{B}_0^\dagger\right)\tilde{c}e^{ik_3\cdot\tilde{X}}\left(\frac{\pi}{4}t\right)\tilde{c}e^{ik_4\cdot\tilde{X}}\left(-\frac{\pi}{4}t\right)\right\rangle \\ &\qquad = (2\pi)^{26}\delta^{26}(\textstyle\sum_i k_i)\pi t \sin\left(\frac{\pi}{2}t\right)^{2\alpha'(k_1\cdot k_2+k_3\cdot k_4)+1}\cos\left(\frac{\pi}{2}t\right)^{2\alpha'(k_1\cdot k_4+k_2\cdot k_3)+1}. \end{aligned} \tag{4.12}$$

Changing variable $y = \sin^2\frac{\pi}{2}t$, we find the four tachyon amplitude as

$$A_4 = \frac{1}{2g^2}(2\pi)^{26}\delta^{26}(\textstyle\sum_i k_i)\displaystyle\int_0^{1/2} dy\, t(y)^{\alpha'\sum k_i^2-4} y^{-\alpha's-\alpha'\sum_i k_i^2/2}(1-y)^{-\alpha'u-\alpha'\sum_i k_i^2/2}, \tag{4.13}$$

where $t(y) = \frac{2}{\pi}\arcsin\sqrt{y}$. Note that the integral appearing in the amplitudes are very simple although we cannot get any analytic expression for generic values of momenta. It is naturally expected that the integral has the divergence caused by the intermediate tachyon.

To get the full tachyon amplitude, we must consider all 4! permutations of external momenta k_i $(i = 1,2,3,4)$. The sum of these 4! permutations give six different terms, and each of these has a factor 4

$$A_4 = \frac{2}{g^2}(2\pi)^{26}\delta^{26}(\textstyle\sum_i k_i)[I(s,u)+I(u,s)+I(u,t)+I(t,u)+I(t,s)+I(s,t)], \tag{4.14}$$

where

$$I(s,u) = \int_0^{1/2} dy\; t(y)^{\alpha'\sum k_i^2-4} y^{-\alpha's-\alpha'\sum_i k_i^2/2}(1-y)^{-\alpha'u-\alpha'\sum_i k_i^2/2}. \tag{4.15}$$

Let us consider the case of on-shell amplitudes where $\alpha' k_i^2 = 1$. Using this on-shell condition, $I(s,u)$ becomes

$$I(s,u) = \int_0^{1/2} dy\; y^{-\alpha' s-2}(1-y)^{-\alpha' u-2}. \tag{4.16}$$

Therefore, $(s \leftrightarrow u)$ term contributes to the range $\frac{1}{2} < y < 1$ after changing variable $y \to 1-y$, so that

$$I(s,u) + I(u,s) = \int_0^{1} dy\; y^{-\alpha' s-2}(1-y)^{-\alpha' u-2} = B(-1-\alpha' s, -1-\alpha' u), \tag{4.17}$$

where $B(a,b)$ is the Euler beta function

$$B(a,b) = \int_0^1 dy\; y^{a-1}(1-y)^{b-1}. \tag{4.18}$$

Other four terms can be combined in the same way. Totally we have the following well-known expression for the tachyon amplitude;

$$A_4 = \frac{2}{g^2}(2\pi)^{26}\delta^{26}(\textstyle\sum_i k_i)[B(-\alpha(s),-\alpha(u)) + B(-\alpha(u),-\alpha(t)) + B(-\alpha(t),-\alpha(s))], \tag{4.19}$$

where $\alpha(s) = 1 + \alpha' s$.

In the case of non-Abelian gauge fields, the procedures are quite similar. However, even in the off-shell amplitude, we have to impose the transversality condition $\epsilon_i \cdot k_i = 0$ on the vector vertex operators $\phi_i = \epsilon_{\mu}{}_i c\partial X^\mu e^{ik_i \cdot X}$ $(i = 1,2,3,4)$, since the formula (4.11) is applicable only to the primary operators.

The four vector amplitude is

$$\begin{aligned} A_4 = \frac{1}{2g^2}(2\pi)^{26}\delta^{26}(\textstyle\sum_i k_i)\Big[& J(s,u)\mathrm{Tr}\,(\lambda^{a_1}\lambda^{a_2}\lambda^{a_3}\lambda^{a_4}) + J(u,s)\mathrm{Tr}\,(\lambda^{a_1}\lambda^{a_4}\lambda^{a_3}\lambda^{a_2}) \\ & + J(u,t)\mathrm{Tr}\,(\lambda^{a_1}\lambda^{a_3}\lambda^{a_2}\lambda^{a_4}) + J(t,u)\mathrm{Tr}\,(\lambda^{a_1}\lambda^{a_4}\lambda^{a_2}\lambda^{a_3}) \\ & + J(t,s)\mathrm{Tr}\,(\lambda^{a_1}\lambda^{a_2}\lambda^{a_4}\lambda^{a_3}) + J(s,t)\mathrm{Tr}\,(\lambda^{a_1}\lambda^{a_3}\lambda^{a_4}\lambda^{a_2})\Big]\mathcal{F}(y;\epsilon,k), \end{aligned} \tag{4.20}$$

where

$$J(s,u) = \int_0^{1/2} dy\; t(y)^{\alpha' \sum k_i^2} y^{-\alpha' s - \alpha' \sum_i k_i^2/2}(1-y)^{-\alpha' u - \alpha' \sum_i k_i^2/2}, \tag{4.21}$$

and the explicit form of function $\mathcal{F}(y;\epsilon,k)$ which includes polarization vectors ϵ_i $(i = 1,2,3,4)$ is given in appendix A.

Imposing on-shell conditions $k_i^2 = 0$, we find

$$\begin{aligned} A_4 = \frac{1}{g^2}(2\pi)^{26}\delta^{26}(\textstyle\sum_i k_i)\int_0^1 dy\; \Big[& y^{-\alpha' s}(1-y)^{-\alpha' u}\,\mathrm{Tr}\,(\lambda^{a_1}\lambda^{a_2}\lambda^{a_3}\lambda^{a_4} + \lambda^{a_1}\lambda^{a_4}\lambda^{a_3}\lambda^{a_2}) \\ & + y^{-\alpha' u}(1-y)^{-\alpha' t}\,\mathrm{Tr}\,(\lambda^{a_1}\lambda^{a_3}\lambda^{a_2}\lambda^{a_4} + \lambda^{a_1}\lambda^{a_4}\lambda^{a_2}\lambda^{a_3}) \\ & + y^{-\alpha' t}(1-y)^{-\alpha' s}\mathrm{Tr}(\lambda^{a_1}\lambda^{a_2}\lambda^{a_4}\lambda^{a_3} + \lambda^{a_1}\lambda^{a_3}\lambda^{a_4}\lambda^{a_2})\Big]\mathcal{F}(y;\epsilon,k). \end{aligned} \tag{4.22}$$

The integral is divergent at the limit $k \to 0$ because of the intermidiate tachyon. After a appropriate regularization, this expression reduces to the four point interactions required for the Yang-Mills action [15].

5. Effective quartic interaction for open superstring

In the previous section, we have shown how the modified use of the Schnabl gauge simplifies the computation of open string amplitudes. In this section, we will extend the analysis to the open superstrings. We are going to derive the formula of the effective quartic coupling for the superstring using WZW-like action [14]

$$S = \frac{1}{4g^2}\Big\langle (e^{-\Phi}Q_B e^{\Phi})(e^{-\Phi}\eta_0 e^{\Phi}) - \int_0^1 dt (e^{-t\Phi}\partial_t e^{t\Phi})\{(e^{-t\Phi}Q_B e^{t\Phi}), (e^{-t\Phi}\eta_0 e^{t\Phi})\}\Big\rangle. \quad (5.1)$$

The ordinary choice of the gauge is

$$b_0\Phi = 0, \quad \xi_0\Phi = 0. \quad (5.2)$$

The cubic terms in this action are extracted as

$$S_3 = \frac{1}{6g^2}\Big[\langle (Q_B\Phi)\Phi(\eta_0\Phi)\rangle - \langle (Q_B\Phi)(\eta_0\Phi)\Phi\rangle\Big]. \quad (5.3)$$

Expanding the field around the background ($\Phi \to \Phi + R$), we get the terms linear in R

$$\begin{aligned}&\langle (Q_B\Phi)R(\eta_0\Phi)\rangle - \langle (Q_B\Phi)(\eta_0\Phi)R\rangle + \langle (Q_B R)\Phi(\eta_0\Phi)\rangle - \langle (Q_B R)(\eta_0\Phi)\Phi\rangle \\ &+\langle (Q_B\Phi)\Phi(\eta_0 R)\rangle - \langle (Q_B\Phi)(\eta_0 R)\Phi\rangle = 3\Big[\langle (Q_B\Phi)R(\eta_0\Phi)\rangle - \langle (Q_B\Phi)(\eta_0\Phi)R\rangle\Big]. \end{aligned} \quad (5.4)$$

Therefore, the total action is

$$\begin{aligned} S &= -\frac{1}{2g^2}\langle \eta_0 R, Q_B R\rangle - \frac{1}{2g^2}\langle R, (Q_B\Phi) * (\eta_0\Phi) + (\eta_0\Phi) * (Q_B\Phi)\rangle + \cdots \\ &= -\frac{1}{2g^2}\langle \eta_0 R, Q_B R\rangle + \frac{1}{2g^2}\langle \eta_0 R, \xi_0\big\{(Q_B\Phi) * (\eta_0\Phi) + (\eta_0\Phi) * (Q_B\Phi)\big\}\rangle + \cdots, \end{aligned} \quad (5.5)$$

where we have used $\xi_0 R = 0$. Shifting the quantum fluctuation field R by

$$R \to R - \frac{1}{2}\frac{b_0}{L_0}\xi_0\big\{(Q_B\Phi) * (\eta_0\Phi) + (\eta_0\Phi) * (Q_B\Phi)\big\} \quad (5.6)$$

to eliminate the terms linear in R, we get the effective quartic coupling

$$S^{(4)} = -\frac{1}{2g^2}\frac{1}{4}\Big\langle Q_B\frac{b_0}{L_0}\xi_0\Phi^{(2)}, \eta_0\frac{b_0}{L_0}\xi_0\Phi^{(2)}\Big\rangle = -\frac{1}{8g^2}\Big\langle \Phi^{(2)}, \frac{b_0}{L_0}\xi_0\Phi^{(2)}\Big\rangle, \quad (5.7)$$

where

$$\Phi^{(2)} = (Q_B\Phi) * (\eta_0\Phi) + (\eta_0\Phi) * (Q_B\Phi). \quad (5.8)$$

In particular, when the on-shell condition $\eta_0 Q_B\Phi = 0$ is satisfied, we can rewrite $\Phi^{(2)}$ as

$$\Phi^{(2)} = -\eta_0\big\{(Q_B\Phi) * \Phi - \Phi * (Q_B\Phi)\big\}. \quad (5.9)$$

Therefore, the effective quartic coupling is given by [15]

$$S^{(4)} = -\frac{1}{8g^2}\Big\langle Q_B\frac{b_0}{L_0}(\Phi * Q_B\Phi - Q_B\Phi * \Phi), \eta_0\frac{b_0}{L_0}(\Phi * Q_B\Phi - Q_B\Phi * \Phi)\Big\rangle. \quad (5.10)$$

JHEP01(2007)011

Again we are going to use $\tilde{z}$ coordinates in the same way as in Witten's cubic action. Gauge conditions (5.2) are transformed by $U_{\tan}$ into

$$\mathcal{B}_0\tilde{\Phi}=0,\quad \tilde{\xi}_0\tilde{\Phi}=0. \tag{5.11}$$

Therefore, states which satisfy eq. (5.2) automatically satisfy these conditions in $\tilde{z}$ coordinates. The first one is what we call the Schnabl gauge condition. We will next consider the modified Schnabl gauge. We define the modified Schnabl gauge for the superstring as

$$\widehat{\mathcal{B}}_0 R=0,\quad \tilde{\xi}_0 R=0. \tag{5.12}$$

Imposing this modified Schnabl gauge for the fluctuation R, the effective quartic term (5.7) is rewritten as

$$S^{(4)}=\frac{-1}{8g^2}\Big\langle (Q_B\tilde{\Phi})*(\eta_0\tilde{\Phi})+(\eta_0\tilde{\Phi})*(Q_B\tilde{\Phi}),\frac{\widehat{\mathcal{B}}_0}{\widehat{\mathcal{L}}_0}\tilde{\xi}_0\big\{(Q_B\tilde{\Phi})*(\eta_0\tilde{\Phi})+(\eta_0\tilde{\Phi})*(Q_B\tilde{\Phi})\big\}\Big\rangle. \tag{5.13}$$

6. Four point amplitude of tachyons and gauge fields

We are now going to compute the four point amplitude of tachyons. In order to deal with GSO($-$) sector that tachyons appear, we consider the string field action for the non-BPS D-brane:

$$S=\frac{1}{4g^2}\Big\langle (e^{-\hat{\Phi}}\hat{Q}_Be^{\hat{\Phi}})(e^{-\hat{\Phi}}\hat{\eta}_0e^{\hat{\Phi}})-\int_0^1 dt(e^{-t\hat{\Phi}}\partial_t e^{t\hat{\Phi}})\{(e^{-t\hat{\Phi}}\hat{Q}_Be^{t\hat{\Phi}}),(e^{-t\hat{\Phi}}\hat{\eta}_0e^{t\hat{\Phi}})\}\Big\rangle, \tag{6.1}$$

In this action, 2×2 internal Chan-Paton factors are added both to the vertex operators and to Q_B and η_0. The tachyon vertex operator is then written as

$$\hat{\phi}=\xi c e^{-\phi}e^{ik\cdot X}\otimes\sigma_1. \tag{6.2}$$

Q_B and η_0 are tensored with σ_3

$$\widehat{Q}_B=Q_B\otimes\sigma_3,\quad \hat{\eta}_0=\eta_0\otimes\sigma_3. \tag{6.3}$$

Since the algebraic structure of this non-BPS action is completely identical to that of BPS action (5.1), we can get the same formula for the quartic coupling as eq. (5.13) up to a factor 2 which compensate the trace of the internal CP matrices. Finally, we find the effective quartic couping

$$S^{(4)}=\frac{-1}{16g^2}\Big\langle (\widehat{Q}_B\tilde{\Phi})*(\hat{\eta}_0\tilde{\Phi})+(\hat{\eta}_0\tilde{\Phi})*(\widehat{Q}_B\tilde{\Phi}),\frac{\widehat{\mathcal{B}}_0}{\widehat{\mathcal{L}}_0}\tilde{\xi}_0\big\{(\widehat{Q}_B\tilde{\Phi})*(\hat{\eta}_0\tilde{\Phi})+(\hat{\eta}_0\tilde{\Phi})*(\widehat{Q}_B\tilde{\Phi})\big\}\Big\rangle. \tag{6.4}$$

Corresponding amplitude is given by the same procedure as in the bosonic string

$$\begin{aligned}
A_4 &= -\frac{1}{16g^2}\int_0^\infty d\beta\Big\langle\Big(I\circ\hat{\phi}_\eta\left(-\frac{\pi}{4}\right)I\circ\hat{\phi}_Q\left(\frac{\pi}{4}\right)+I\circ\hat{\phi}_Q\left(-\frac{\pi}{4}\right)I\circ\hat{\phi}_\eta\left(\frac{\pi}{4}\right)\Big)\\
&\quad\times\widehat{\mathcal{B}}_0\widehat{U}_{2\beta+4}\tilde{\xi}_0\Big(\hat{\phi}_Q\left(\frac{\pi}{4}\right)\hat{\phi}_\eta\left(-\frac{\pi}{4}\right)+\hat{\phi}_\eta\left(\frac{\pi}{4}\right)\hat{\phi}_Q\left(-\frac{\pi}{4}\right)\Big)\Big\rangle\\
&=\frac{-1}{16g^2}\int_0^{1/2}dt\; t^{\sum h_i-1}\Big\langle\Big(I\circ\hat{\phi}_\eta\left(-\frac{\pi}{4}t\right)I\circ\hat{\phi}_Q\left(\frac{\pi}{4}t\right)+I\circ\hat{\phi}_Q\left(-\frac{\pi}{4}t\right)I\circ\hat{\phi}_\eta\left(\frac{\pi}{4}t\right)\Big)\\
&\quad\times\widehat{\mathcal{B}}_0\tilde{\xi}_0\Big(\hat{\phi}_Q\left(\frac{\pi}{4}t\right)\hat{\phi}_\eta\left(-\frac{\pi}{4}t\right)+\hat{\phi}_\eta\left(\frac{\pi}{4}t\right)\hat{\phi}_Q\left(-\frac{\pi}{4}t\right)\Big)\Big\rangle,
\end{aligned} \tag{6.5}$$

where we have defined

$$\begin{aligned}\hat{\phi}_Q &= (Q_B \otimes \sigma_3)(\tilde{\phi} \otimes \sigma_1)\\ &= \left[\left(-\alpha' k^2 + \frac{1}{2}\right)\partial\tilde{c}\tilde{c}\tilde{\xi}e^{-\tilde{\phi}}e^{ik\cdot\tilde{X}} + \sqrt{2\alpha'}k^\mu\tilde{\psi}_\mu\tilde{c}e^{ik\cdot\tilde{X}} - \tilde{\eta}e^{\tilde{\phi}}e^{ik\cdot\tilde{X}}\right]\otimes i\sigma_2, \qquad (6.6)\\ \hat{\phi}_\eta &= (\eta_0 \otimes \sigma_3)(\tilde{\phi}\otimes\sigma_1)\\ &= \tilde{c}e^{-\tilde{\phi}}e^{ik\cdot\tilde{X}} \otimes i\sigma_2. \qquad (6.7)\end{aligned}$$

Evaluating all correlation functions in eq. (6.5), we get

$$\begin{aligned}A_4 = &-\frac{2}{16g^2}\int_0^{1/2} dt\; t^{\sum(\alpha' k_i^2-\frac{1}{2})-1}(2\pi)^d\delta^d(\textstyle\sum k_i)y^{-\alpha' s-\alpha'\sum k_i^2/2}(1-y)^{-\alpha' t-\alpha'\sum k_i^2/2}\\ &\times\Bigg[\eta_{\mu\nu}\pi t\sin\left(\frac{\pi}{2}t\right)\cos\left(\frac{\pi}{2}t\right)\left(2\alpha'(k_1^\mu k_3^\nu + k_2^\mu k_4^\nu) - 2\alpha'(k_1^\mu k_4^\nu + k_2^\mu k_3^\nu)\frac{1}{\cos^2\frac{\pi}{2}t}\right)\\ &+\left(-\alpha' k_1^2+\frac{1}{2}\right)\left(\frac{1}{2}\cot\left(\frac{\pi}{4}t\right)(\pi t\cos\pi t-\sin\pi t)+\frac{1}{2}\cot\left(\frac{\pi}{4}t\right)\sec\left(\frac{\pi}{2}t\right)(\pi t-\sin\pi t)\right)\\ &-\left(-\alpha' k_3^2+\frac{1}{2}\right)\left(\frac{1}{8}\pi t(4\cos\pi t-\sin\pi t)\tan\left(\frac{\pi}{4}t\right)-\frac{1}{8}\pi t\sec\left(\frac{\pi}{2}t\right)(4+\sin\pi t)\tan\left(\frac{\pi}{4}t\right)\right)\\ &-\left(-\alpha' k_4^2+\frac{1}{2}\right)\left(\frac{1}{8}\pi t\sec\left(\frac{\pi}{2}t\right)(-4+\sin\pi t)\tan\left(\frac{\pi}{4}t\right)+\frac{1}{8}\pi t(4\cos\pi t+\sin\pi t)\tan\left(\frac{\pi}{4}t\right)\right)\\ &+\left(-\alpha' k_2^2+\frac{1}{2}\right)\left(\frac{1}{2}\cot\left(\frac{\pi}{4}t\right)\sec\left(\frac{\pi}{2}t\right)(\pi t\cos\pi t-\sin\pi t)+\frac{1}{2}\cot\left(\frac{\pi}{4}t\right)(\pi t\cos\pi t-\sin\pi t)\right)\Bigg].\end{aligned}\qquad (6.8)$$

It was shown that the WZW-like action reproduces the on-shell four point amplitudes correctly [16]. Here we will see the consistency of the above amplitude at on-shell. Imposing $\alpha' k_i^2 = 1/2$ $(i = 1,2,3,4)$, we find that on-shell four tachyon amplitude is obtained as

$$\begin{aligned}A_4 = &\frac{-1}{4g^2}(2\pi)^d\delta^d(\textstyle\sum k_i)\displaystyle\int_0^{1/2} dy\Big[-y^{-\alpha' s-1}(1-y)^{-\alpha' u-2}(\alpha' u+1)\\ &+y^{-\alpha' s-1}(1-y)^{-\alpha' u-1}(\alpha' t+1)\Big].\end{aligned}\qquad (6.9)$$

We rewrite the first term of eq. (6.9) by partial integration

$$\begin{aligned}\int_0^{1/2} dy\; y^{-\alpha' s-1}(1-y)^{-\alpha' u-2}(\alpha' u+1) &= \int_0^{1/2} dy\; y^{-\alpha' s-1}\partial_y(1-y)^{-\alpha' u-1}\\ &= (1/2)^{-\alpha' s-1}(1/2)^{-\alpha' u-1}\\ &\quad+(\alpha' s+1)\int_0^{1/2} dy\; y^{-\alpha' s-2}(1-y)^{-\alpha' u-1}.\end{aligned}\qquad (6.10)$$

Summation over 4! permutations of the momenta yields to Euler beta functions. Using identity

$$
\begin{aligned}
-(-\alpha' s-1)B(-\alpha' s-1,-\alpha' u) &= -(-\alpha' s-1)\frac{\Gamma(-\alpha' s-1)\Gamma(-\alpha' u)}{\Gamma(-\alpha' s-\alpha' u-1)} \\
&= \frac{-\Gamma(-\alpha' s)\Gamma(-\alpha' u)}{\Gamma(-\alpha' s-\alpha' u)}(-\alpha' s-\alpha' u-1) \\
&= B(-\alpha' s,-\alpha' u)(\alpha' t+1),
\end{aligned}
\tag{6.11}
$$

we find

$$
\begin{aligned}
A_4 = \frac{-1}{g^2}(2\pi)^d\delta^d(\textstyle\sum k_i)\Big[&2(1+\alpha' u)B(-\alpha' s,-\alpha' t)-(1/2)^{-\alpha' s-1}(1/2)^{-\alpha' t-1} \\
&+2(1+\alpha' t)B(-\alpha' s,-\alpha' u)-(1/2)^{-\alpha' s-1}(1/2)^{-\alpha' u-1} \\
&+2(1+\alpha' s)B(-\alpha' t,-\alpha' u)-(1/2)^{-\alpha' t-1}(1/2)^{-\alpha' u-1}\Big].
\end{aligned}
\tag{6.12}
$$

In order to get the complete four point amplitude, we have to consider summation over the permutations of momenta. In addition, since the superstring field action (6.1) is non-polynomial, quartic coupling which describes contact interaction of four string fields

$$
S_4 = \frac{1}{4!\cdot 2g^2}\Big[-2\langle(\widehat{Q}_B\widehat{\Phi})\widehat{\Phi}(\hat{\eta}_0\widehat{\Phi})\widehat{\Phi}\rangle + \langle(\widehat{Q}_B\widehat{\Phi})\widehat{\Phi}^2(\hat{\eta}_0\widehat{\Phi})\rangle + \langle(\widehat{Q}_B\widehat{\Phi})(\hat{\eta}_0\widehat{\Phi})\widehat{\Phi}^2\rangle\Big], \tag{6.13}
$$

also contributes to the four point amplitudes. We expect that the on-shell four point tachyon amplitude in total agrees with the first quantization result [17]. The contributions from the contact interaction (6.13) is given by

$$
\begin{aligned}
A_4^{\text{contact}} = -\frac{1}{g^2}(2\pi)^d\delta^d(\textstyle\sum k_i)\Big[&(1/2)^{-\alpha' s-1}(1/2)^{-\alpha' u-1}+(1/2)^{-\alpha' u-1}(1/2)^{-\alpha' t-1} \\
&+(1/2)^{-\alpha' t-1}(1/2)^{-\alpha' s-1}\Big].
\end{aligned}
\tag{6.14}
$$

This contribution just cancels the extra terms in eq. (6.12) and the result agrees with that of the first quantization [17]

$$
\begin{aligned}
A_4 + A_4^{\text{contact}} = \frac{-2}{g^2}(2\pi)^d\delta^d(\textstyle\sum k_i)\Big[&(1+\alpha' u)B(-\alpha' s,-\alpha' t) \\
&+(1+\alpha' t)B(-\alpha' s,-\alpha' u)+(1+\alpha' s)B(-\alpha' t,-\alpha' u)\Big].
\end{aligned}
\tag{6.15}
$$

Four point amplitude of gauge fields is obtained in the similar way. For simplicity, we only consider the gauge fields with zero momenta. The vertex operator is given by

$$
\phi = \tilde{c}\tilde{\xi}e^{-\tilde{\phi}}\tilde{\psi}^\mu. \tag{6.16}
$$

From eq. (5.10) on-shell four point amplitude is given by

$$
\begin{aligned}
A_4 &= -\frac{1}{8g^2}\Big\langle(\phi*\phi_Q-\phi_Q*\phi),\eta_0\frac{\widehat{\mathcal{B}}_0}{\widehat{\mathcal{L}}_0}(\phi*\phi_Q-\phi_Q*\phi)\Big\rangle \\
&= -\frac{1}{8g^2}\int_0^{1/2}\frac{dt}{t}\Big\langle\Big(I\circ\phi_Q\left(-\frac{\pi}{4}t\right)I\circ\phi\left(\frac{\pi}{4}t\right)-I\circ\phi\left(-\frac{\pi}{4}t\right)I\circ\phi_Q\left(\frac{\pi}{4}t\right)\Big) \\
&\quad\times\eta_0\widehat{\mathcal{B}}_0\Big(\phi\left(\frac{\pi}{4}t\right)\phi_Q\left(-\frac{\pi}{4}t\right)-\phi_Q\left(\frac{\pi}{4}t\right)\phi\left(-\frac{\pi}{4}t\right)\Big)\Big\rangle,
\end{aligned}
\tag{6.17}
$$

where

$$\begin{aligned}\phi_Q &= Q_B\phi \\ &= i\sqrt{\frac{2}{\alpha'}}\tilde{c}\partial\tilde{X}^\mu. \end{aligned} \tag{6.18}$$

After the evaluation of this correlation functions, integration can be done explicitly

$$\begin{aligned} A_4 &= \frac{1}{8g^2}\int_0^{1/2} dy\, \left(2\eta^{\nu\sigma}\eta^{\mu\rho} + 2\frac{\eta^{\nu\rho}\eta^{\mu\sigma}}{(1-y)^2}\right) \\ &= \frac{1}{4g^2}\left(\frac{1}{2}\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}\right). \end{aligned} \tag{6.19}$$

Contribution from eq. (6.13) is

$$A_4^{\text{contact}} = \frac{1}{g^2}\left(\frac{1}{8}\eta^{\mu\rho}\eta^{\nu\sigma} - \frac{1}{2}\eta^{\mu\sigma}\eta^{\nu\rho}\right). \tag{6.20}$$

Then the total four point amplitude is just the Yang-Mills quartic coupling.

$$A_4 = \frac{1}{g^2}\left(\frac{1}{4}\eta^{\mu\rho}\eta^{\nu\sigma} - \frac{1}{4}\eta^{\mu\sigma}\eta^{\nu\rho}\right), \tag{6.21}$$

which was first pointed out in [15] via the Siegel gauge.

7. Discussions

We have obtained the formula for the four point amplitudes in $\tilde{z}$ coordinates. Even in the Schnabl gauge, the off-shell amplitudes are very complicated. In this paper, we proposed the use of the modified version of the Schnabl gauge and showed that this gives the simple formula of the four point off-shell amplitudes. If one wants to calculate the amplitudes in this gauge with this formula, the states also have to satisfy $\widehat{\mathcal{B}}_0$ condition. However, the states we used do not satisfy this modified gauge condition since these are fixed in the Siegel gauge. This mixed gauge choice simplify the off-shell amplitudes and is useful to calculate the on-shell amplitudes. In fact the on-shell four point amplitudes for tachyons and photons are reproduced correctly. This method is applicable for the open superstring field theory.

More interesting question is whether the Schnabl gauge is effective for obtaining closed string amplitudes. In the closed string field theory, the calculations of the amplitudes are so difficult even at on-shell. It would be interesting to investigate whether the amplitudes of the closed string fields could be obtained in the Schnabl gauge.

We have postponed the arguments about the physical meaning of modified use of the Schnabl gauge in the external fields. It is not clear whether this condition fixes the gauge uniquely. At the linearized level, it might be shown that this gauge condition is consistent by the method used for the Siegel gauge.

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A. Kinematical factor $\mathcal{F}$

In this appendix, we list the kinetic factor which appeared in eq. (4.20).

$$
\begin{aligned}
\mathcal{F}(y;\epsilon,k) =\ & \epsilon_1\cdot\epsilon_2\epsilon_3\cdot\epsilon_4 y^{-2} + \epsilon_1\cdot\epsilon_3\epsilon_2\cdot\epsilon_4 + \epsilon_1\cdot\epsilon_4\epsilon_2\cdot\epsilon_3(1-y)^2 \\
& -2\alpha'\Big[\epsilon_1\cdot\epsilon_2(k_2\cdot\epsilon_3)\Big\{(k_2\cdot\epsilon_4)(1-y)^{-1} + (k_3\cdot\epsilon_4)y^{-1}(1-y)^{-1}\Big\} \\
& -\epsilon_1\cdot\epsilon_2(k_4\cdot\epsilon_3)\Big\{(k_2\cdot\epsilon_4)y^{-1} + (k_3\cdot\epsilon_4)y^{-2}\Big\} \\
& +\epsilon_1\cdot\epsilon_3(k_3\cdot\epsilon_2)\Big\{(k_3\cdot\epsilon_4)y^{-1}(1-y)^{-1} + (k_2\cdot\epsilon_4)(1-y)^{-1}\Big\} \\
& -\epsilon_1\cdot\epsilon_3(k_4\cdot\epsilon_2)\Big\{(k_3\cdot\epsilon_4)y^{-1} + (k_2\cdot\epsilon_4)\Big\} \\
& -\epsilon_1\cdot\epsilon_4(k_4\cdot\epsilon_3)\Big\{(k_4\cdot\epsilon_2)y^{-1} + (k_3\cdot\epsilon_2)y^{-1}(1-y)^{-1}\Big\} \\
& +\epsilon_1\cdot\epsilon_4(k_2\cdot\epsilon_3)\Big\{(k_4\cdot\epsilon_2)(1-y)^{-1} + (k_3\cdot\epsilon_2)(1-y)^{-2}\Big\} \\
& -\epsilon_2\cdot\epsilon_3(k_2\cdot\epsilon_1)\Big\{(k_2\cdot\epsilon_4)y^{-1} + (k_1\cdot\epsilon_4)y^{-1}(1-y)^{-1}\Big\} \\
& +\epsilon_2\cdot\epsilon_3(k_4\cdot\epsilon_1)\Big\{(k_2\cdot\epsilon_4)(1-y)^{-1} + (k_1\cdot\epsilon_4)(1-y)^{-2}\Big\} \\
& +\epsilon_2\cdot\epsilon_4(k_2\cdot\epsilon_3)\Big\{(k_2\cdot\epsilon_1)y^{-1}(1-y)^{-1} + (k_3\cdot\epsilon_1)(1-y)^{-1}\Big\} \\
& +\epsilon_2\cdot\epsilon_4(k_1\cdot\epsilon_3)\Big\{(k_2\cdot\epsilon_1)y^{-1} + (k_3\cdot\epsilon_1)\Big\} \\
& -\epsilon_3\cdot\epsilon_4(k_3\cdot\epsilon_2)\Big\{(k_3\cdot\epsilon_1)(1-y)^{-1} + (k_2\cdot\epsilon_1)y^{-1}(1-y)^{-1}\Big\} \\
& +\epsilon_3\cdot\epsilon_4(k_1\cdot\epsilon_2)\Big\{(k_3\cdot\epsilon_1)y^{-1} + (k_2\cdot\epsilon_1)y^{-2}\Big\}\Big] \\
& +4\alpha'^2\Big\{-\epsilon_1\cdot k_3\epsilon_2\cdot k_4\epsilon_3\cdot k_2\epsilon_4\cdot k_2 + \epsilon_1\cdot k_3\epsilon_2\cdot k_4\epsilon_3\cdot k_4\epsilon_4\cdot k_2(1-y)y^{-1} \\
& +\epsilon_1\cdot k_3\epsilon_2\cdot k_3\epsilon_3\cdot k_4\epsilon_4\cdot k_2 y^{-1} + \epsilon_1\cdot k_4\epsilon_2\cdot k_4\epsilon_3\cdot k_4\epsilon_4\cdot k_2 y^{-1} \\
& -\epsilon_1\cdot k_3\epsilon_2\cdot k_4\epsilon_3\cdot k_2\epsilon_4\cdot k_3 y^{-1} + \epsilon_1\cdot k_3\epsilon_2\cdot k_4\epsilon_3\cdot k_4\epsilon_4\cdot k_3(1-y)y^{-2} \\
& +\epsilon_1\cdot k_3\epsilon_2\cdot k_3\epsilon_3\cdot k_4\epsilon_4\cdot k_3 y^{-2} + \epsilon_1\cdot k_4\epsilon_2\cdot k_4\epsilon_3\cdot k_4\epsilon_4\cdot k_3 y^{-2} \\
& -\epsilon_1\cdot k_3\epsilon_2\cdot k_3\epsilon_3\cdot k_2\epsilon_4\cdot k_2(1-y)^{-1} - \epsilon_1\cdot k_4\epsilon_2\cdot k_4\epsilon_3\cdot k_2\epsilon_4\cdot k_2(1-y)^{-1} \\
& +\epsilon_1\cdot k_4\epsilon_2\cdot k_3\epsilon_3\cdot k_4\epsilon_4\cdot k_2 y^{-1}(1-y)^{-1} - \epsilon_1\cdot k_3\epsilon_2\cdot k_3\epsilon_3\cdot k_2\epsilon_4\cdot k_3 y^{-1}(1-y)^{-1} \\
& -\epsilon_1\cdot k_4\epsilon_2\cdot k_4\epsilon_3\cdot k_2\epsilon_4\cdot k_3 y^{-1}(1-y)^{-1} + \epsilon_1\cdot k_4\epsilon_2\cdot k_3\epsilon_3\cdot k_4\epsilon_4\cdot k_3 y^{-2}(1-y)^{-1} \\
& -\epsilon_1\cdot k_4\epsilon_2\cdot k_3\epsilon_3\cdot k_2\epsilon_4\cdot k_2(1-y)^{-2} - \epsilon_1\cdot k_4\epsilon_2\cdot k_3\epsilon_3\cdot k_2\epsilon_4\cdot k_3 y^{-1}(1-y)^{-2}\Big\}.
\end{aligned}
\tag{A.1}
$$

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